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Incentives and Stability in Some Two-Sided Economic and Social Models

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Abstract

The stability and incentives aspects of problems involving matching subsets of two disjoint sets of agents by allocating bundles of goods or social alternatives is discussed. It is proved that the set of stable allocations, or the core, of any such problem is never empty and that each set of agents has its own attainable optimal stable allocation. Then it is demonstrated that although in general it is impossible to construct a stable allocation mechanism which confines the problem of manipulation to one set of agents, for a large class of problems, termed the one-on-one matching problems, it is possible to construct an allocation mechanism which always yields stable allocations and which prevents any of the agents in at least one of the two sets from benefitting from manipulation.

1. Introduction

The importance of economic models concerning the relations between two disjoint sets of agents was recognized by many, as Shapley and Shubik (1972) observe: "Two-sided market models are important, as Cournot, Edgeworth, Böhm-Bawerk, and others have observed, not only for the insights they may give into more general economic situations with many types of traders, consumers, and producers, but also for the simple reason that in real life many markets and most actual transactions are in fact bilateral - i.e., bring together a buyer and a seller of a single commodity."

This work is a study of some aspects common to many two-sided economic and social problems, particularly the class of generalized matching problems: problems involving matching subsets of two disjoint sets of agents by allocating bundles of goods or social alternatives.

Some of these are simple matching problems--matching men with women, students with the colleges of their choice, etc. Some are more complex--matching firms with workers, where a firm may hire the same worker at any one of various wage levels; or matching sellers, each selling a single item in simultaneous sales, with buyers who may participate in more than one sale. Similarly, consider producers of the same product who are located in different markets and the problem of matching them with consumers and a price per item which may differ from market to market, etc.

This study is mainly concerned with the possibility of constructing allocation mechanisms which will generate for all such problems stable

outcomes, and which will give agents the incentive to honestly reveal their preferences over the possible outcomes.

The results derived here build on and extend a number of previous contributions. Gale and Shapley (1962) while studying the simple matching problem derived two remarkable results. They proved that for any such problem there is always at least one stable match. (A match is stable if no two agents from opposite sets prefer each other to their current partners. Shapley and Shubik (1972) pointed out that the set of stable matches is equal to the core of the cooperative game resulting when any two agents in opposite groups can create a match if they both agree.) They further proved that each of the two sets of agents has its own optimal match among the stable matches, and the two are not necessarily identical. Optimal means that every member of the same set (weakly) prefers this match to any other stable match. They also constructed a matching procedure which always chooses the optimal match of one of the sets of agents. Crawford and Knoer (1981) extended both these results to the case of job matching with heterogeneous firms and workers under conditions of perfect information, when a firm and a potential employee may negotiate for wages.

Dubins and Freedman (1981) and Roth (1982) recognized that once a matching procedure is adopted for a simple matching problem, it becomes a noncooperative game among the agents whose strategy choices are their reported preferences over their potential partners. They studied the possibility of constructing matching procedures that will always choose stable matches and will also give incentives to all the participating agents to reveal their true preferences, i.e., that the true preferences

should be dominant strategies. Roth (1982) proved that no matching procedure exists which always yields a stable match and gives all agents the incentive to honestly reveal their preferences, even though procedures exist which accomplish either of these goals separately. However, Dubins and Freedman (1981) and Roth (1982) independently proved the surprising result that matching procedures do exist which always yield the optimal match of one of the groups of agents and which give incentives to all agents in this group to honestly report their preferences. Dubins and Freedman (1981) additionally proved that these procedures are such that no coalition of agents in this group exists, whose members can benefit by simultaneously misrepresenting their preferences.

This discussion concerns the cases of general matching problems which involve a finite number of agents, each with a strict preference over finite sets of alternatives. First, it will be shown that the set of stable outcomes, or the core, of any such problem is never empty and that each set of agents has its own attainable optimal stable outcome. The last is again a somewhat surprising result since, e.g., it implies that all competing buyers, in a problem of matching sellers of single items with buyers, will prefer the same stable pairing of buyers, sellers and items, to all other stable market allocations.

Then it will be demonstrated that in general it is impossible to create an allocation procedure which always yields a stable outcome and which gives all agents in one group incentives to honestly reveal their true preferences. Nevertheless it will be proved that the Dubins and Freedman (1981) and Roth (1982) results are extendable to a large class of problems which I term the class of one-on-one matching problems. This

class contains the problems in which every agent is matched with at most one agent of the opposite set and with a bundle of alternatives. Thus, e.g., the simple matching problems, the job matching problems, and the problems of matching sellers with unique buyers, are all one-on-one matching problems. It will be proved for this class of problems that allocation procedures exist which always select the optimal outcome of one set of agents and for which no coalition of agents from this set exists whose members can benefit by simultaneously misrepresenting their preferences.

The formal model is described in section 2, the stability aspect is investigated in section 3, the results on stability and incentives are derived in section 4 and their significance is discussed in section 5.

2. Formulation

Let W and F be two disjoint sets of m and n agents respectively. For every $i \in W$ and $j \in F$ let A_i and B_j be finite nonempty sets of alternatives available to agents i and j respectively. Every set of alternatives A_i contains a null alternative a_o^i which stands for the option of "i does not participate." Similarly every B_j contains a null alternative b_o^j. A <u>social outcome</u> or <u>group allocation</u> is an (n+m) - vector $X = (x_1,\ldots,x_m,y_1,\ldots,y_n) \in A_1 \times \ldots \times A_m \times B_1 \times \ldots \times B_n$. An outcome X is a <u>feasible outcome</u> if for every $i \in W$, $x_i \in \{y_1,\ldots,y_n\} \cup \{a_o^i\}$ and for every $j \in F$, $y_j \in \{x_1,\ldots,x_m\} \cup \{b_o^j\}$. Let S denote the set of all feasible social outcomes.

For $i \in W$ and $j \in F$ let Σ_i and Ω_j be the sets of all antisymmetric, transitive, and complete binary relations on A_i and B_j respectively. An element $p \in \Sigma_i (q \in \Omega_j)$ represents the preferences of individual i (j)

over his set of alternatives. Thus xp_iy means agent i strictly prefers alternative x to alternative y etc. Agent $i \varepsilon W$ ($j \varepsilon F$) is said to prefer social outcome X to social outcome X' if he prefers x_i to x_i' (y_j to y_j'). Thus agents may not be indifferent between alternatives, but they are indifferent between any two social outcomes which allocate to them the same alternative. A profile of preferences P is an (n+m) - vector $P = (p_1,\ldots,p_m,q_1,\ldots,q_n) \varepsilon \Sigma_1 \times\ldots\times \Sigma_m \times \Omega_1 \times\ldots\times \Omega_n$. A generalized matching problem is the problem of associating profiles with feasible social outcomes.

Notice that the above definitions imply that if an agent prefers alternative x to alternative y, then he prefers every feasible group allocation that allocates to him x to every feasible group allocation that allocates to him alternative y. Thus any problem in which agents can take into consideration their preferences over coalitions of agents that share with them a group allocation, is ruled out of the scope of this paper (see example 6).

All problems discussed here from now on are assumed to be generalized matching problems. A generalized matching problem is a one-on-one matching problem if every feasible social outcome $X \varepsilon S$ is such that for every $i \varepsilon W$, x_i is either a_o^i or there exists exactly one $j \varepsilon F$ such that $y_j = x_i$, and for every $j \varepsilon F$ either $y_j = b_o^j$ or there exists exactly one $i \varepsilon W$ such that $x_i = y_j$. An allocation procedure G is a function $G: \Sigma_1 \times \ldots \times \Sigma_m \times \Omega_1 \times \ldots \times \Omega_n \rightarrow S$, which chooses a feasible outcome for any profile of preferences. Thus allocation procedures are the institutional arrangements by which a generalized matching problem is solved.

To illustrate some of the above notations and some of the problems included in the scope of this work, consider the following examples.

Example 1. A simple matching problem (A marriage problem).

Let $W = \{1,2\}$ and $F = \{\bar{1},\bar{2},\bar{3}\}$ be sets of men and women respectively. Then for $i=1,2$, $A_i = \{(i,\bar{1}),(i,\bar{2}),(i,\bar{3}),a_o^i\}$ and for $j=\bar{1},\bar{2},\bar{3}$, $B_j = \{(1,j),(2,j),b_o^j\}$ where an ordered pair (i,j) stands for--man i married to woman j. Thus, $((1,\bar{1}),(2,\bar{2}),(1,\bar{1}),(2,\bar{2}),b_o^3)$ is an example of a feasible outcome which indicates man 1 married to woman $\bar{1}$, man 2 married to woman $\bar{2}$, and woman $\bar{3}$ is not married.

Example 2. Job matching.

Let $W = \{1,2,3\}$ and $F = \{\bar{1},\bar{2}\}$ be sets of workers and firms respectively. Assume each firm has one job opening and it can hire each worker at one of two different wage levels. Then for $i \in W$, $A_i = \{x(i,\bar{1}),y(i,\bar{1}),x(i,\bar{2}),y(i,\bar{2}),a_o^i\}$ and for $j \in F$, $B_j = \{x(1,j),y(1,j),x(2,j),y(2,j),x(3,j),y(3,j),b_o^j\}$ where $x(i,j)$ stands for--'worker i hired by firm j at the lower level of wages', while $y(i,j)$ stands for--'worker i hired by firm j at the higher level of wages'. A feasible outcome for example, is $(x(1,\bar{2}),a_o,y(3,\bar{1}),y(3,\bar{1}),x(1,\bar{2}))$ which states that the first worker is hired by the second firm at the lower level of salary, the third worker is hired by the first firm at the higher level, while the second worker is unemployed.

Example 3. Simultaneous sales

Let $W = \{1,2,3\}$ and $F = \{\bar{1},\bar{2}\}$ be sets of buyers and sellers respectively. Assume that each seller is selling a single item, each buyer is endowed with a finite amount of indivisible goods that can be

traded for the auctioned items, and may participate in both sales; and that each buyer may buy at most, one item. In this case $A_1 = \{a_o^1, (a_1,j), \ldots, (a_k,j) \text{ for } j=\bar{1},\bar{2}\}$ where (a_ℓ,j) stands for--'a_ℓ traded for the j^{th} item', $A_2 = \{a_o^2, (b_1,j), \ldots, (b_\ell,j) \text{ for } j=\bar{1},\bar{2}\}$ and $A_3 = \{a_o^3, (c_1,j), \ldots, (c_2,j) \text{ for } j=\bar{1},\bar{2}\}$ while for $j \varepsilon F$, $B_j = \{(a_1,j), \ldots, (a_k,j), (b_1,j), \ldots, (b_\ell,j), (c_1,j), \ldots, (c_2,j), b_o^j\}$. Thus an example of a feasible outcome is $(a_o^1, (b_3,2), (c_2,1), (c_2,1), (b_3,2))$ which translates into: The second buyer bought the second item 'paying' b_3 while the third buyer bought and paid c_2 for the first item.

All the above are examples of one-on-one matching problems. The following two are examples of generalized matching problems which are not one-on-one problems.

Example 4. Multiple markets of a single product.

Let $F = \{\bar{1},\bar{2},\bar{3}\}$ be a set of three different producers of the same product, each located at a different market, and let $W = \{1,2,3,4,5\}$ be a set of five consumers for this product. Assume that producers and consumers negotiate for the price per item, in such manner that all consumers who buy from the same producer, pay the same price. In this case for $i \varepsilon W$, $A_i = \{a_o^i, (p_1,j), \ldots, (p_k,j) \text{ for } j \varepsilon F\}$ and for $j \varepsilon F$, $B_j = \{(p_1,j), \ldots, (p_k,j), b_o^i\}$ where p_1 to p_k is a finite list of possible prices and (p_ℓ,j) indicates that price p_ℓ is paid in market j. Then an example of a feasible outcome is $((p_3,2), (p_3,2), (p_2,1), (p_5,3), a_o^5, (p_2,1), (p_3,2), (p_5,3))$ indicating that consumers one and two buy from the second producer paying p_3 per item, the third consumer buys from the first producer paying p_2, the fourth

consumer buys from the third producer paying p_5 per item, while the last consumer decided not to participate.

Example 5.

Consider a department which is trying to fill up to three open positions at the junior level, and which has already identified three candidates, all of whom are identically qualified. The only obstacle to signing the three is the department's rule that all starting employees will agree to the same package of benefits—which may include a combination of salary, retirement payment, insurance, working load, etc. The department is indifferent between hiring one, two or all three.

Let $F = \{\bar{1}\}$, $W = \{1,2,3\}$ represent the department and the candidates respectively. Let $A = \{a,b,c\}$ be a set of three available packages. Then for $i \in W$, $A_i = A \cup \{a_o^i\}$ and $B_{\bar{1}} = A \cup \{b_o^1\}$, and e.g., a feasible outcome is (b,b,a_o^3,b) which stands for: the first and second candidates agreed to package b, while the third candidate decided to go somewhere else.

The following example is the case of a matching problem which is not even a generalized matching problem.

Example 6. Multiple matches.

Let F and W be sets of universities and candidates for faculty positions respectively. Suppose the number of candidates a university may hire depends on the combination of candidates which it can attract. E.g., suppose the first university would like to hire candidates 1, 2 and 3. But if this is not possible it would rather hire candidates 4

and 5 than any other combination of candidates 1, 2 and 3. This is a problem that cannot be stated as a generalized matching problem.

3. Stability

A social outcome $\bar{x}$ is said to be unstable given profile P if any of the following three conditions exists: (i) there exists an agent $k \in W \cup F$ such that $a_o^k p_k \bar{x}_k$; (ii) there exist $i \in W$, $j \in F$ and alternative $x \in A_i \cap B_j$ such that $x p_i \bar{x}_i$ and $x q_j \bar{y}_i$; or (iii) there exist $i \in W$, $j \in F$ and alternative $x \in A_i \cap B_j$ such that either $x = \bar{x}_i$ and $x q_j \bar{y}_j$ or $x p_i \bar{x}_i$ and $x = \bar{y}_j$. An outcome is stable given P if it is not unstable.[1)] This terminology is motivated by the assumption that agents cannot be compelled to accept an outcome, thus any agent can elect not to participate, if he prefers this to what is allocated to him and any two agents of opposite sets can create a match if there is an alternative which both prefer to their current allocation. It is easy to verify that the set of stable outcomes is the core of the cooperative game that results under these assumptions.

A stable outcome is W-optimal (F-optimal) if every agent in W (in F) (weakly) prefers it to any other stable outcome.

An allocation procedure G is a stable allocation procedure if for every profile P, the outcome G(P) is a stable outcome. The following two theorems are generalizations of the results derived by Gale and Shapley (1962) for the simple matching problem and the results derived by Crawford and Knoer (1981) for the job matching problem in a discrete market (theorems 1 and 3). The proofs are modifications of Gale and Shapley's original discussions.

<u>Theorem 1</u>

The stable outcomes set of any generalized matching problem is never empty.

<u>Proof</u>:

The following is a description of an allocation procedure G_o which is a generalization of the Gale and Shapley (1962) assignment procedure. It will be proved that this procedure is a stable allocation procedure. (Notice that although the procedure is described with W and F "playing" different roles, the algorithm doesn't lose its generality, since W and F are interchangeable. The only difference is that a different stable outcome may be chosen.)

<u>Step 0</u>: Every agent $j \in F$ is assigned the null alternative b_o^j.

<u>Step 1</u>: a. Every $i \in W$ offers his most preferred alternative (if different from a_o^i) to all agents in F, for whom it is feasible.

b. Each agent $j \in F$ rejects all but his most preferred alternative among those suggested to him, including b_o^j.

⋮

<u>Step k</u>: a. Every agent $i \in W$ who has been "rejected" in step k-1, offers his most preferred alternative among those not yet rejected (if different from a_o^i) to all the agents in F for whom it is feasible (an agent is said to be <u>rejected</u> if <u>all</u> the agents in F to whom he offered his alternative rejected it). If his most preferred alternative is a_o^i he ceases to participate in the process.

b. Each agent $j \in F$ rejects all but his most preferred alternative among those alternatives offered to him in step k and the alternative he kept from step k-1.

The procedure terminates when no more agents are rejected

Since the numbers of agents in both W and F are finite and since all alternative sets are also finite, G_o must terminate with a feasible social outcome in a finite number of steps.

Suppose there exists a profile of preferences $\overline{P}$ such that $\overline{X} = G_o(\overline{P})$ is not a stable outcome. This indicates that there is a feasible social outcome $\hat{X}$ and there are agents $i \in W$ and $j \in F$ such that $\hat{x}_i p_i \overline{x}_i$, $\hat{y}_j q_j \overline{y}_j$ and $\hat{x}_i = \hat{y}_j$ (notice that the way the procedure is defined, $\overline{X}$ cannot be unstable due to an agent's preference not to participate rather than accept his $\overline{X}$ allocation nor due to condition (iii) in the definition of unstable allocations). $\hat{x}_i p_i \overline{x}_i$ implies that i offered $\hat{x}_i$ to individual j (possibly to other agents in F as well). Since $\overline{y}_j \neq \hat{x}_i$ it implies that agent j rejected $\hat{x}_i$ at one stage, hence $\overline{y}_j q_j \hat{x}_i$, a contradiction. This proves that G_o is stable.

Q.E.D.

Theorem 2

There is always a stable outcome which is W-optimal and a stable outcome which is F-optimal.

Proof:

It will be proved that the outcome of the G_o procedure is always the optimal outcome for W, the set of agents which offer the alternatives. Thus by exchanging the roles of W and F, the theorem can be fully proved.

The proof is by induction. An alternative x is possible for $(i,j) \varepsilon W \times F$ (given P) if there exists a stable outcome $\bar{X}$ such that $\bar{x}_i = x = \bar{y}_j$. An alternative x is possible for $i \varepsilon W$ if there exists $j \varepsilon F$ such that x is possible for (i,j). Suppose that up to step k-1 in the G_o procedure, no possible outcome for an agent in W is rejected, and that at step k agent $i \varepsilon W$ is rejected when offering alternative $\bar{x}$, i.e., that every agent $j \varepsilon F$ to whom $\bar{x}$ was offered rejected it either because he prefers b_o^j to $\bar{x}$ or because alternative y, which he prefers to $\bar{x}$, was offered to him. If j prefers b_o^j to $\bar{x}$, obviously $\bar{x}$ is not possible for (i,j). On the other hand, if y, such that $yq_j\bar{x}$, was offered to j by $\ell \varepsilon W$ (among others perhaps), this indicates that ℓ prefers y to any (other) possible alternative for himself (this by the induction assumption) and therefore any social outcome that allocates alternative $\bar{x}$ to i and j is unstable given P (because of ℓ, j and y), which means that $\bar{x}$ is not possible for (i,j). Since this is true for every $j \varepsilon F$ that $\bar{x}$ was offered to, it means that $\bar{x}$ is not possible for i. By induction, this proves that no possible alternative for agents in W is ever

rejected in the G_o procedure. Therefore for any given profile P, $G_o(P)$ is (weakly) preferred by all agents in W to any other possible outcome, hence $G_o(P)$ is W-optimal.

Q.E.D.

4. Incentives and Stability

An allocation procedure is said to be manipulable if there exists an agent k, either in W or in F, a profile of preferences P and $\hat{p}_k$ such that if $X = G(P)$ and $\hat{X} = G(P_{-k}|\hat{p}_k)$ then $\hat{x}_k p_k x_k$. ($P_{-k}|\hat{p}_k$ stands for the profile derived from the profile P where $\hat{p}_k$ replaces p_k.) An allocation procedure is a straightforward procedure (or incentive compatible) if it is not manipulable: if none of the agents can benefit by misrepresenting his true preferences on his alternatives. Thus no agent has the incentive not to honestly reveal his preferences. To put it differently, an allocation procedure is straightforward if in the resulting noncooperative game among the agents (where strategy choices are preferences and payoffs are the outcomes), revealing his true preferences is a dominant strategy for every participating agent. An allocation procedure G is W-group (F-group) manipulable if there exist a coalition $K \subset W$ ($K \subset F$), a profile P and $\hat{p}_k$ ($\hat{q}_k$) for every $k \in K$, such that if $X = G(P)$ and $\hat{X} = G(P_{-K}|\hat{P}_K)$ then $\hat{x}_k p_k x_k$ ($\hat{y}_k q_k y_k$) for every $k \in K$.

An allocation is W-straightforward (F-straightforward) if it is not W-group (F-group) manipulable.

In this section we investigate the possibility of constructing stable and straightforward procedures for the generalized matching problem. The focus of interest in stable allocation mechanisms is that the

outcomes they generate are always Pareto-efficient and they eliminate the need to compel the participating agents to accept these outcomes; once a stable allocation is chosen, no coalition of agents can do better by disregarding the allocation procedure. The appeal of straightforward procedures is that they minimize the amount of information agents need to accumulate and report for the procedure to operate properly. Actually all an agent needs to know is his own true preferences. This is especially important when combined with additional requirements, such as the requirement for efficient or stable outcomes. Although the resulting outcomes are efficient and/or stable when agents report truthfully, the outcomes may not possess these properties when agents misrepresent their preferences. The following two theorems are immediate extensions of results reported in Roth (1981), therefore the proof of theorem 3 is omitted and the proof of theorem 4 follows closely his discussion.

The first theorem states that it is impossible to construct a straightforward allocation procedure which always yields a stable outcome for the generalized matching problem, while the second theorem demonstrates that there do exist efficient straightforward mechanisms which are not stable.[2)]

Theorem 3:

No stable straightforward allocation procedures exist for all generalized matching problems.

Once the requirement of stability is dropped, it is easy to construct a Pareto-efficient and straightforward allocation procedure by implementing a lexicographically dictatorial rule.

Theorem 4:

There exist Pareto-efficient and straightforward procedures for the generalized matching problems.

Proof:

Consider the following procedure.

For any profile P of preferences:

Step 1: Agent 1 in W is allocated his most preferred alternative.

⋮

Step k: Agent $k \in W$ gets his most preferred alternative among those still feasible for him (an alternative x is still feasible for agent k if either $x = a_o^k$ or if $x \in A_k \cap B_j$ for some $j \in F$ and either x is already allocated to j (because of some agent $i \in F$, $i < k$) or j is still free.)

⋮

Step m+1: All agents in F which are still with no allocation are allocated their null alternative. This is a lexicographically dictatorial allocation procedure which is therefore a straightforward procedure and clearly its outcomes are always Pareto-efficient.

Q.E.D.

Dubins and Freedman (1981) and Roth (1982) demonstrated that by exploiting the special two-sided structure of the simple matching problem, the difficulties with manipulability can be confined to one group of agents.

They proved that the matching procedure which always chooses the optimal match of one set of agents, say the W set, is a stable W-straightforward procedure.[3)]

The following theorem demonstrates that in general it is impossible to confine all the difficulties with manipulability to one set of agents.

Theorem 5:

No stable allocation procedure for all generalized matching problems is also W-straightforward (F-straightforward).

Proof:

The proof is by example. Let $W = \{1,2\}$, $F = \{\bar{1}\}$, $A = \{a,b,c\}$, $A_1 = A \cup \{a_o^1\}$, $A_2 = A \cup \{a_o^2\}$ and $B_{\bar{1}} = A \cup \{b_o^1\}$. Suppose G is a stable allocation procedure. Let $p_1, p_1' \in \Sigma_1$, $p_2, p_2' \in \Sigma_2$, $q_1 \in \Omega_{\bar{1}}$ be as follows: $ap_1bp_1cp_1a_o^1$, $bp_1'a_o^1p_1'ap_1'c$, $bp_2cp_2ap_2a_o^2$, $bp_2'a_o^2p_2'cp_2'a$ and $cq_1aq_1bqb_o^1$. Since the set of stable allocations for (p_1,p_2,q_1) is $\{(c,c,c)\}$, for (p_1',p_2,q_1) is $(b,b,b),(a_o^1,c,c)\}$ and for (p_1',p_2',q_1) it is $\{(b,b,b)\}$, therefore it must be $G((p_1,p_2,q_1)) = (c,c,c)$ and $G((p_1',p_2',q_1)) = (b,b,b)$. If $G((p_1',p_2,q_1)) = (b,b,b)$, then G is manipulable by agent 1, since bp_1c, while if $G((p_1',p_2,q_1)) = (a_o^1,c,c)$, then G is manipulable by agent 2 since $bp_2cp_2a_o^2$. Hence no stable allocation procedure is also W-straightforward for this problem.

Q.E.D.

The above theorem rules out the possibility of constructing any stable allocation procedure, let alone a W or F optimal procedure, for all the generalized matching problems which also limit the possibility

of manipulation to one set of agents. Nevertheless as the following theorem will demonstrate, there is a large class of generalized matching problems, for which it is possible to construct W (or F) optimal allocation procedures, the class of one-on-one matching problems, which eliminate the possibility of manipulations by agents in W (or F).

To prove the main result the following two lemmas are needed.

A coalition $K \subset W$ $(K \subset F)$ is said to <u>manipulate simply</u> if there exists P, $\hat{P} = (P_{-K}, \hat{P}_K)$, $X = G(P)$ and $\hat{X} = G(\hat{P})$, such that for every $k \in K$, $\hat{x}_k p_k x_k$ and $\hat{x}_k \hat{p}_k y$ for every other $y \in A_k$.

<u>Lemma 1</u>:[4)]

If a coalition $K \subset W$ $(K \subset F)$ can manipulate G_o, it can manipulate it simply.

<u>Proof</u>:

Without loss of generality assume $K = \{1,2,...,k\} \subset W$ and X, $\hat{X}$, P, and $\hat{P}$ are such that $\hat{P} = P_{-K}|\hat{P}_K$, $X = G_o(P)$, $\hat{X} = G_o(\hat{P})$ and for every $i \in K$ $\hat{x}_i p_i x_i$. Let $\tilde{P} = P_{-K}|\tilde{P}_K$ where for every $i \in K$, $\tilde{p}_i \in \Sigma_i$ is such that $\hat{x}_i \tilde{p}_i y$ for every other $y \in A_i$.

Observe that $\hat{X}$ is stable given $\tilde{P}$. If not, then there would exist $i \in W$, $j \in F$ and $x \in A_i \cap B_j$ such that $x \tilde{p}_i \hat{x}_i$ and $x \tilde{q}_j \hat{y}_i$. This implies that $i \in K$ (if $i \notin K$ then $\tilde{p}_i = \hat{p}_i$ and $\tilde{q}_j = \hat{q}_j$ which implies that $\hat{X}$ is unstable in $\hat{P}$) but by assumption $\hat{x}_i \tilde{p}_i y$ for every $y \in A_i$, a contradiction. Let $\tilde{X} = G_o(\tilde{P})$, then by theorem 2, $\tilde{X}$ is W-optimal in $\tilde{P}$, hence it must be $\tilde{x}_i = \hat{x}_i$ for every $i \in K$, which completes the proof that K can manipulate simply.

Q.E.D.

Lemma 2:

If a social outcome $\hat{X}$ is the result of simple manipulation of G_o, by a coalition in W (in F), then every agent in W (in F) (weakly) prefers his allocation in $\hat{X}$ to his allocation in X--the outcome before manipulation.

Proof:

Suppose this is not the case. Then there exists coalition $K \subset W$, profiles P and $\tilde{P}$ and social outcomes X and $\tilde{X}$ such that $X = G_o(P)$, $\tilde{X} = G_o(\tilde{P})$, $\tilde{P}$ is the profile generated when every agent in K manipulates simply, and for at least one agent $i \in W$, $i \notin K$, $x_i p_i \hat{x}_i$. This implies that when G_o is operating on $\tilde{P}$, at least one agent in W offered an alternative he did not offer when G_o operated on P, to at least one agent in F. Let $\ell \in W$ be one of the first agents to do so (all of them during the same step in G_o) and let this alternative be $y \in A_\ell$. Obviously $\ell \notin K$, hence $\tilde{p}_\ell = p_\ell$ and therefore $x_\ell p_\ell y$. This implies that every agent $j \in F$ who accepted x_ℓ in X (i.e., $y_j = x_\ell$), rejects x_ℓ because a better alternative is offered to him now, but then this must be an alternative that wasn't offered to him in the case of P, which contradicts the assumption that ℓ was one of the first agents to offer a new alternative.

Q.E.D.

Theorem 6:

The allocation procedure which always selects the W-optimal (F-optimal) outcome is a W-straightforward (F-straightforward) procedure for all one-on-one-matching problems.

Proof:

Notice that the theorem is phrased in terms of "The allocation procedure...". There are obviously different procedures which select the same outcomes. From the point of view of manipulation all such procedures are equivalent, and can be regarded as a single allocation procedure.

The proof is by induction. Suppose there exists a coalition $K \subset W$ which can manipulate the G_o procedure for a one-on-one matching problem. Then, by lemma 1, this coalition can manipulate simply and in this case, by lemma 2, every agent in W (weakly) prefers the alternative allocated to him in the second case to the alternative allocated to him in the first case (I will refer to the case of no manipulation as the first case and to the case of simple manipulation as the second case). This implies that no agent in W offered an alternative in the second case which he didn't offer in the first case. Suppose the number of steps which took place in the execution of the G_o procedure in the first case is t. Observe that all agents in F which accepted alternatives in step t, actually accepted the first offer they preferred to "not participating". Since no agent in W offers a new alternative in the second case and since in every one-on-one matching problem the number of active participants is the same in both groups of agents--we must conclude that every agent in W which offered an alternative to an agent in F during step t in the first case, must offer the same alternative to the same agent in the second case, hence he doesn't belong to K. Suppose every agent in W that reached his final alternative, in the first case, between steps k+1 and t ends up with the same alternative (and the same

agent in F), and therefore does not belong to K. Suppose $i_o \varepsilon W$ stops at step k, sharing alternative y_o with agent $j_o \varepsilon F$. Let D be the set of all alternatives rejected by j_o, in the first case, and suppose alternative z is his most preferred alternative in D. If $z \neq b_o^{j_o}$, then it was offered to j_o by some agent $i_1 \varepsilon W$, which indicates that i_1 reaches his final allocation after step k, hence by the assumption reaches the same alternative in the second case. This implies that agent i_1 offers z to agent j_o also in the second case, which indicates that alternative y_o must be offered to j_o by i_o, (since this is the only alternative offered to j_o which he prefers to z, and no new alternatives were offered) hence i_o and j_o share the alternative y_o in the second case too, or - $i_o \not\varepsilon K$. If $z = b_o^{j_o}$, it indicates that i_o was the first and last agent in W to offer an alternative to j_o which he prefers to $b_o^{j_o}$ in the first case. Since j_o must participate in the second case too, (because of the one-on-one condition) and since no new alternatives are offered in the second case, it again implies that i_o and j_o are sharing y_o in the second case, hence $i_o \not\varepsilon K$. By induction this proves that K is an empty coalition, a contradiction.

Q.E.D.

<u>Corollary 1</u> (Dubins and Freedman (1981))

The Gale-Shapley assignment procedure is a stable W-straightforward (F-straightforward) procedure for the simple matching problems.

Consider the job matching problem, introduced in example 2, of matching workers with firms which may hire each worker at one of different wage levels. By extending the notations introduced in example 2,

it is easy to demonstrate that this is a one-on-one matching problem thus:

Corollary 2:

The G_o allocation procedure is W-straightforward for the job matching problems.

Example 3 introduces the problem of selecting unique buyers for a number of simultaneous sales of single items. Again, by extending the notation introduced there, to the general case, it is easy to prove that this is also a one-on-one matching problem.

Corollary 3:

The G_o allocation procedure is a W-straightforward procedure for the simultaneous sales problem.

The above corollaries emphasize a number of one-on-one matching problems for which it is possible to confine the difficulties with manipulations to one group of agents. Thus implementation of the G_o procedure in each of these problems will create the institutional mechanism which will always choose a stable allocation as an outcome and which will eliminate the possibility of strategic behavior by at least one group of agents. E.g., implementing the G_o procedure as the method by which buyers bid for items in the simultaneous sales problem, will cause the buyers' optimal allocation always to be the market outcome--and will give all the buyers the incentive always to reveal their true preferences.

5. Conclusions

It was demonstrated that the two-sided structure of the models discussed here, allows for some important and surprising results. Theorem 1 proves the existence of stable allocations for every generalized matching problem. Theorem 2 proves that each group of agents has a stable outcome which all its members prefer to any other stable allocation—which is a counter intuitive conclusion since agents in the same group compete with each other and are not expected to agree on a most preferred final outcome. Theorem 2 also suggests ways to reach these "optimal points" since implementation of the G_o procedure can serve as the institutional mechanism which will yield these allocations.

It was also proved that for all one-on-one matching problems these procedures restrict the temptation of manipulation to one group of agents only. Thus the difficulties with manipulation can be eliminated from problems where one group is either a group of regulated agents or public institutions.

Footnotes

1. The weak dominance definition (see Roth and Postlewaite (1977)) is used here in order to avoid stable allocations which are not Pareto optimal.

2. Roth's (1982) impossibility result is of the same nature as the Gibbard and Satterthwaite impossibility results for constructing Pareto-efficient, non-dictatorial and straightforward social choice rules. The impossibility of constructing such social choice rules (with additional assumptions on the domains of preferences) was proved in a number of works (e.g., see Gibbard (1973), Satterthwaite (1975), Kalai and Muller (1977), Maskin (1976) and Ritz (1981) for analysis of the different cases).

3. Roth (1982) proved that it is W-straightforward for all "coalitions" of one agent.

4. Roth (1982) uses similar lemmas to prove his theorem 5. By proving that it is impossible that every man will find a better match, he proves the impossibility of simple manipulation and thus of any kind of manipulation (by a single man) for the simple matching problem.

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